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Schwinger and Pegg–Barnett approaches and a relationship between angular and Cartesian quantum descriptions: II. Phase spaces

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Abstract
Following the discussion—in state-space language—presented in a preceding paper, we work on the passage from the phase-space description of a degree of freedom described by a finite number of states (without classical counterpart) to one described by an infinite (and continuously labelled) number of states. With this it is possible to relate an original Schwinger idea to the Pegg–Barnett approach to the phase problem. In phase-space language, this discussion shows that one can obtain the Weyl–Wigner formalism, for both Cartesian *and* angular coordinates, as limiting elements of the discrete phase-space formalism.

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1. Introduction

In a previous work it was shown that the usual quantum descriptions of Cartesian and angular coordinates in state space can both be seen as different limiting cases of the Schwinger programme of treating quantum discrete variables [1]. The limiting process involved reproduces the Pegg–Barnett (PB) approach to phase variables in the case of angle/angular momentum variables [2]. The purpose of this work is to translate that discussion into a phase-space point of view, which might be a way to unify under the same structure three apparently different formalisms, each one adapted to one specific kind of quantum variable, namely Cartesian, angular and discrete. In doing so, we again relate the PB and Schwinger approaches, now through the phase-space representatives of number and phase operators.

The phase-space picture is a well established picture of quantum mechanics [3–10], specially if one deals with degrees of freedom with classical counterparts, a situation in which the Weyl–Wigner formalism is the undisputed approach. Nevertheless, to cope with variables such as rotation angle and angular momentum, the formalism had to be adapted in order to account for the inherent periodicity involved. This was accomplished in the late 1970s [11, 12] and developed to its full extent in [13]. However, when one deals with degrees of freedom *without* classical counterparts, the formulations discussed above are not applicable. In these cases, there is a formalism completely capable of dealing with the peculiarities of the finite/discrete character of the variables [14–16], in much the same spirit as the Weyl–Wigner formalism itself. Following the procedure shown in [1] which led from discrete variables to the cases with classical counterparts, we show that the usual Weyl–Wigner formalism and *also* its rotation angle–angular momentum version naturally emerges from the discrete phase-space formalism. In addition, some properties of the discrete Wigner function are also discussed. From a more rigorous mathematical point of view, these limiting processes presented in this context have been also discussed in [17–19].

This paper is organized as follows. In section 2 we briefly present the main ideas of the discrete phase-space representation, drawing attention to some properties of the discrete Wigner function, while in section 3 we discuss the limiting processes which lead the original operator bases to the well known Weyl–Wigner continuous case as well as the particular case of rotations. Finally, section 4 is devoted to the conclusions.

2. Discrete phase space

As has already been shown [14–16], a discrete phase-space representation of a quantum mechanical degree of freedom which is characterized by a finite number of states, and therefore with no classical counterpart, can be established if we are given a basis in the corresponding operator space. One such basis was introduced by Schwinger in his seminal paper on this subject [20], constructing it from some particular unitary cyclically shifting pairs of operators, and another has been proposed that basically considers the double Fourier transform of that of Schwinger [14]. As previously shown, once we are provided with such an operator basis, it is a direct task to obtain the discrete phase-space representatives of the operators acting on the state space from which we started. To briefly summarize these results let us consider the operator basis and recall its main properties.

The discrete phase-space formalism is set over the basis elements

$$G(j,l) = \frac{1}{N}\sum_{m,n=-h}^{h} U^m V^n \exp\left(\frac{\mathrm{i}\pi mn}{N}\right)\exp\left[-\frac{2\pi\mathrm{i}}{N}(mj+nl)\right]\exp[\mathrm{i}\pi\phi(m+h,n+h;N)], \tag{1}$$

where $(j,l) \in [-h,h]$, $h = \frac{N-1}{2}$ (for simplicity, odd N values will be considered, as even values only require only a little more care and a heavier notation). The modular phase $\phi(m,n;N)$, included to warrant an explicit mod N symmetry in the summing indices of the basis, is given by

$$\phi(m,n;N) = N I_m^N I_n^N - m I_n^N - n I_m^N \tag{2}$$

with

$$I_k^N = \left[\frac{k}{N}\right] \tag{3}$$

standing for the integral part of k with respect to N. The Us and Vs are the Schwinger unitary operators [20], briefly reviewed in [1].

As a basis, the set (1) can be used to represent all linear operators acting on the given N-dimensional state space; this can be accomplished by a direct decomposition

$$\hat{O} = \sum_{m,n=0}^{N-1} O(m,n)G(m,n), \tag{4}$$

where the coefficient, $O(m,n)$, that gives rise to the representative of the operator $\hat{O}$ in the discrete phase space [14], is given by

$$O(m,n) = \frac{1}{N}\,\mathrm{Tr}[G(m,n)\hat{O}], \tag{5}$$

where we used the fact that $G(m,n)$ is self-adjoint.

The basic properties of the basis, equation (1), are:

(1) $\mathrm{Tr}[G(m,n)] = 1;$ (6)

(2) $\mathrm{Tr}[G^\dagger(m,n)G(r,s)] = N\delta^{[N]}_{m,r}\delta^{[N]}_{n,s};$ (7)

(3)
$$[\mathrm{Tr}\, G^\dagger(m,n)G(u,v)G(r,s)] = \sum_{a,b,c,d=-h}^{h} \frac{1}{N^2}\exp\left[\frac{\mathrm{i}\pi}{N}(bc-ad)\right] \times \exp[-\mathrm{i}\pi\phi(a+c+h, b+d+h; N)] \times \exp\left\{\frac{2\pi\mathrm{i}}{N}[a(m-u)+b(n-v)+c(m-r)+d(n-s)]\right\}, \tag{8}$$

where the last expression is important for the mapping of products of operators [21]. Particular interest resides in the mapping of the commutator of two operators, for then it is possible to study, for example, the time evolution of the density operator in the von Neumann–Liouville equation [16, 22].

2.1. The discrete Wigner function

The phase-space representative of the density operator in the discrete approach is also referred to as the (discrete) Wigner function [14,23,24]. If the (pure) state of a given system is described by

$$|\psi\rangle = \sum_n \psi_n |u_n\rangle, \tag{9}$$

where $\{|u_n\rangle\}$ is the (complete and orthonormal) set of eigenvectors of the Schwinger operator U, then the use of equation (5) leads to a Wigner function of the form

$$\rho_w(m,n) = \frac{1}{N^2}\sum_{j,l,k} \psi_k^*\psi_{k-l}\exp\left[\frac{2\pi\mathrm{i}}{N}\left(jk - \frac{jl}{2} - mj - nl\right)\right], \tag{10}$$

or

$$\rho_w(m,n) = \frac{1}{N^2}\sum_{l,k}\psi_k^*\psi_{k-l}\frac{\sin[\pi(k-m-l/2)]}{\sin[\frac{\pi}{N}(k-m-l/2)]}\exp\left[-\frac{2\pi i}{N}nl\right].$$

Its main properties are, in direct analogy with the usual continuous Wigner function:

(1) It is a real function, as follows from the Hermicity of the basis elements.

(2) Summing it over each one of its indices gives the probability distribution in the other. For example,

$$\sum_n \rho_w(m,n) = \sum_n \frac{1}{N^2}\sum_{j,l,k} \psi_k^*\psi_{k-l}\exp\left[\frac{2\pi \mathrm{i}}{N}\left(jk - \frac{jl}{2} - mj - nl\right)\right], \tag{11}$$

such that

$$\sum_n \rho_w(m,n) = \frac{1}{N}\sum_{j,l,k} \psi_k^*\psi_{k-l}\exp\left[\frac{2\pi \mathrm{i}}{N}\left(jk - \frac{jl}{2} - mj\right)\right]\delta_{l,0}^{[N]}, \tag{12}$$

and so

$$\sum_n \rho_w(m,n) = |\psi_m|^2. \tag{13}$$

In the same way, the summation over $\{m\}$ would lead to the probability distribution associated with the eigenstates of the Schwinger operator V.

(3) It must be different from zero at at least N sites in the discrete phase space. Writing it as

$$\rho_w(m,n) = \frac{1}{N^2}\sum_{j,l}\exp\left[-\frac{2\pi \mathrm{i}}{N}(mj+nl)\right]\sum_k \psi_k^*\psi_{k-l}\exp\left[\frac{2\pi \mathrm{i}}{N}\left(jk - \frac{jl}{2}\right)\right], \tag{14}$$

it is clear that it is the double Fourier transform of the quantity $\rho_s(j,l)$,

$$\rho_s(j,l) = \sum_k \psi_k^*\psi_{k-l}\exp\left[\frac{2\pi \mathrm{i}}{N}\left(jk - \frac{jl}{2}\right)\right], \tag{15}$$

which, in its turn, can be seen as the inner product of two vectors $\{\psi_k \exp[-\frac{2\pi \mathrm{i}}{N} jk]\}$ and $\{\psi_{k-l}\exp[-\frac{\pi \mathrm{i}}{N} jl]\}$ of unit length. By the Schwarz inequality it is clear than that $|\rho_s(j,l)|^2 \leqslant 1$, and from properties of the discrete Fourier transform one can also conclude that

$$(\rho_w(m,n))^2 \leqslant 1. \tag{16}$$

Now, using the property [16]

$$\mathrm{Tr}[\hat{O}_1\hat{O}_2] = \frac{1}{N}\sum_{m,n} O_1(m,n)O_2(m,n), \tag{17}$$

then

$$\mathrm{Tr}[(|\psi\rangle\langle\psi|)^2] = \frac{1}{N}\sum_{m,n}(\rho_w(m,n))^2, \tag{18}$$

which leads to

$$1 = \frac{1}{N}\sum_{m,n}(\rho_w(m,n))^2, \tag{19}$$

and considering inequality (16) we conclude that the discrete Wigner function must be different from zero at N sites at least in the discrete phase space.

3. The continuum limit in phase space

The continuum limit of an operator representative in phase space is to be seen as its behaviour in the infinite-dimensional/continuum limit. We now follow a procedure similar to that of [1].

3.1. Cartesian coordinates

We start from the discrete space operator basis elements,

$$G(j,l) = \frac{1}{N}\sum_{m,n=-h}^{h} U^m V^n \exp\left(\frac{i\pi mn}{N}\right)\exp\left[-\frac{2\pi i}{N}(mj+nl)\right], \tag{20}$$

where we omit the modular phase since we shall restrict ourselves to sums in the interval $[-h, h]$. Then we introduce the scaling parameter

$$\epsilon = \sqrt{\frac{2\pi}{N}}, \tag{21}$$

which will become infinitesimal as $N \to \infty$. We also introduce two Hermitian operators $\{P, Q\}$,

$$P = \sum_{j=-\frac{N-1}{2}}^{\frac{N-1}{2}} j\epsilon^{\delta} p_0 |v_j\rangle\langle v_j| \qquad Q = \sum_{j'=-\frac{N-1}{2}}^{\frac{N-1}{2}} j'\epsilon^{2-\delta} q_0 |u_{j'}\rangle\langle u_{j'}|, \tag{22}$$

constructed from the projectors of the eigenstates of U and V. Again, δ is a free parameter which might assume any value in the open interval $(0, 2)$. $\{p_0, q_0\}$ are real parameters that might carry units of momentum and position, respectively, and $\epsilon^{\delta} p_0$ and $\epsilon^{2-\delta} q_0$ are the distance between successive eigenvalues of the P and Q operators. With the help of these, we can rewrite the Schwinger operators as

$$V = \exp\left[\frac{i\epsilon^{2-\delta}P}{p_0}\right] \qquad U = \exp\left[\frac{i\epsilon^{\delta}Q}{q_0}\right]. \tag{23}$$

and perform the change of variables

$$\begin{array}{ll} q = q_0\epsilon^{2-\delta} j & p = p_0\epsilon^{\delta} l \\ u = p_0\epsilon^{\delta} m & v = -q_0\epsilon^{2-\delta} n. \end{array} \tag{24}$$

With this, we arrive at new operator basis elements that do not explicitly depend on δ; however, at the same time, the operators U and V carry a particular ϵ dependence, defined by the particular choice of δ, namely

$$G(p,q) = \frac{1}{q_0 p_0 \epsilon^2 N}\sum_{u,v=-h}^{h} \Delta u \Delta v \exp\left[\frac{iuQ}{p_0q_0}\right]\exp\left[-\frac{ivP}{p_0q_0}\right]\exp\left(-\frac{i}{2p_0q_0}uv\right) \times \exp\left[-\frac{i}{p_0q_0}(qu - pv)\right]. \tag{25}$$

If we take the limit $N \to \infty$, it is clear that we can consider $\Delta u \to du$ and $\Delta v \to dv$, yielding

$$G(p,q) = \frac{1}{2\pi q_0 p_0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} du\, dv \exp\left[\frac{iu(Q-q-v/2)}{p_0q_0}\right]\exp\left[-\frac{iv(P-p)}{p_0q_0}\right]. \tag{26}$$

As we know from [1] that in this limit we recover the usual results for position and momentum once $p_0q_0 = \hbar$, we use the identity

$$|q\rangle\langle q| = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dx \exp\left[\frac{ix(Q-q)}{\hbar}\right], \tag{27}$$

and obtain

$$G(p,q) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dv\, |q+v/2\rangle\langle q+v/2| \exp\left[-\frac{iv(P-p)}{\hbar}\right] \tag{28}$$

$$G(p,q) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dv\, |q+v/2\rangle\langle q-v/2| \exp\left[\frac{ivp}{\hbar}\right], \tag{29}$$

which is exactly the form of the Weyl–Wigner basis elements $\Delta(p, q)$. It is interesting to see that, as in the state-space description, the parameter δ does not affect the final result since, in this case (any $\delta \in (0, 2)$), the basis elements do not depend on it at all, although U and V do. It is now a trivial matter to prove that the decomposition coefficients are well behaved in the limit and also go to the Weyl–Wigner coefficients. From this we see that the whole mapping scheme is recovered. This result has already been achieved for the particular case $\delta = 1$ in [16], where the limiting process which leads to the Moyal bracket was also discussed. Moreover, it has to be stressed that, starting from the continuous family of unitary operators, equation (23), and realizing the independence of the basis elements from δ, the Weyl–Wigner basis elements are overdetermined in the limiting process, since, for any $\delta \in (0, 2)$ pair of operators, we always obtain the same final expression. This means that for the continuous family of unitary operators (except for $\delta = 0$ or 2), as proposed, the continuum limit is the Weyl–Wigner operator basis.

From these results one immediately concludes that the discrete Wigner function has the ordinary Wigner function as its continuum limit, in the sense discussed above. As we have already stated, most properties of the usual Wigner function are originally present in the discrete one, and emerge as the continuum limit of the latter.

In the discrete case we have seen that the Wigner function must be different from zero at N sites at least in phase space. It is obvious that the same procedure which led to this result would lead to the well known property of the usual Wigner function that it must be different from zero in a region of the phase space of area at least $\hbar$. This discussion illustrates somewhat quantitatively how the quantum effects become more and more drastic as the dimensionality N decreases.

3.2. Angular coordinates

Following our analogy with what was done in [1], we now choose the parameter δ in the extreme situation $\delta = 0$. We expect now to obtain a phase-space formalism which is consistent with angular coordinates. We start again from our discrete operator space basis elements, equation (20),

$$G(j', l') = \frac{1}{N} \sum_{m',n'=-h}^{h} U^{m'} V^{n'} \exp\left(\frac{\mathrm{i}\pi m'n'}{N}\right) \exp\left[-\frac{2\pi \mathrm{i}}{N}(m'j' + n'l')\right].$$

Rewriting the Schwinger operators as above, but with $\delta = 0$, we now would have

$$M = \sum_{j=-\frac{N-1}{2}}^{\frac{N-1}{2}} j m_0 |v_j\rangle\langle v_j| \qquad \text{and} \qquad \Theta = \sum_{j'=-\frac{N-1}{2}}^{\frac{N-1}{2}} j' \epsilon^2 \theta_0 |u_{j'}\rangle\langle u_{j'}|, \tag{30}$$

leading to

$$V = \exp\left[\frac{\mathrm{i}\epsilon^2 M}{m_0}\right] \qquad \text{and} \qquad U = \exp\left[\frac{\mathrm{i}\Theta}{\theta_0}\right], \tag{31}$$

so that only V now depends on ϵ, and changing the variables as

$$\begin{aligned} \theta &= \theta_0 \epsilon^2 j' \qquad & l &= l_0 l' \\ m &= m_0 m' \qquad & \alpha &= -\theta_0 \epsilon^2 n' \end{aligned} \tag{32}$$

we have for the basis elements

$$G(\theta, l) = -\frac{1}{2\pi\theta_0} \sum_{m=-m_0 h}^{m_0 h} \sum_{\alpha=(\pi-\frac{\pi}{N})\theta_0}^{(-\pi+\frac{\pi}{N})\theta_0} \Delta\alpha \exp\left[\frac{\mathrm{i}m\Theta}{m_0\theta_0}\right] \exp\left[-\frac{\mathrm{i}\alpha M}{m_0\theta_0}\right]$$
$$\times \exp\left(-\frac{\mathrm{i}m\alpha}{2m_0\theta_0}\right) \exp\left[-\frac{\mathrm{i}}{m_0\theta_0}(m\theta - l\alpha)\right]. \tag{33}$$

Performing again the limit $N \to \infty$, the angle variables become continuous and we have

$$G(\theta, l) = \frac{1}{2\pi\theta_0} \sum_{m=-\infty}^{\infty} \int_{-\pi\theta_0}^{\pi\theta_0} \mathrm{d}\alpha \exp\left[\mathrm{i}m\left(\Theta - \theta - \frac{\alpha}{2}\right)\right] \exp\left[-\frac{\mathrm{i}\alpha(M-l)}{m_0\theta_0}\right]. \quad (34)$$

The sum over m is the projector in angle space (θ_0 is set to unity, so the angle units are radians, and $m_0\theta_0$ is set to $\hbar$), and

$$G(\theta, l) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\alpha \left|\theta + \frac{\alpha}{2}\right\rangle\left\langle\theta + \frac{\alpha}{2}\right| \exp\left[-\frac{\mathrm{i}\alpha(M-l)}{\hbar}\right], \quad (35)$$

so that, with the use of equation (30), we achieve the result

$$G(\theta, l) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\alpha \left|\theta + \frac{\alpha}{2}\right\rangle\left\langle\theta - \frac{\alpha}{2}\right| \exp\left[\frac{\mathrm{i}l\alpha}{\hbar}\right], \quad (36)$$

that is precisely the result of [12, 13]. We remark that we have no need to worry about the periodicities in the angle variable as our angular states are bounded to the $[-\pi, \pi)$ interval by *definition*, and our notation has mod N periodicity (mod 2π in the continuum limit) by construction [1]. It would seem at first glance that the continuum interval is $[-\pi, \pi]$, but this is not the case, as it can be seen from the original discrete results that the states in the extremes of the interval are *not* the same. We understand that, once the basis elements are recovered, the whole mapping procedure is recovered.

Again, all properties of the angular Wigner function can be obtained from its discrete counterpart by the limiting process above. It must be stated however that in many cases it turns out to be easier to work with the discrete rather than the angular case. That is particularly true in the obtention of the angular counterpart of equation (16), which in the angular case does not lead to a condition involving a minimal area unit in phase space due to the very nature of the angular phase space.

It is interesting to note that what were considered to be *conditions* for the existence of the Wigner function in [12, 13] are derived as properties of it in the present scheme.

3.3. Mapping of the Pegg–Barnett operators

The number and phase operators of PB can be immediately mapped on the discrete phase space. In fact, we exactly reproduce the PB scheme if we rename the M operator of equation (31) as N and include a reference angle in the definition of Θ (which must be an integer multiple of $\frac{2\pi}{N}$). The phase-space representatives of these operators, through direct use of equation (5), are seen to be

$$N(m, n) = n, \qquad \Theta(m, n) = \theta_{ref} + \frac{2\pi}{N}m, \quad (37)$$

with obvious continuum limits.

4. Conclusions

Motivated by the results of part I, we looked for a phase-space discussion of the limits which connect discrete, angular and Cartesian coordinates. It then became clear that the Weyl–Wigner formalism, in both position–momentum *and* angle–angular momentum cases, can be seen as limiting elements of a discrete phase-space formalism. The angle–angular momentum case is seen to be in deep connection with the PB approach to the phase problem, while the Weyl–Wigner operator basis is reobtained for all the cases for which the parameter governing the unitary operators is different from zero; in this sense the Weyl–Wigner basis is overdetermined in the limiting process. An interesting by-product of this discussion is the analysis of the Wigner function, which reproduced the conditions imposed on the angular Wigner function in [12, 13].

With all this in mind, one is compelled to regard this as a kind of standard, or rather 'natural' approach to phase space in quantum mechanics. The basic feature that pertains to all three versions of the formalism is that one constructs a basis in operator space from the Fourier transform of the shifting operators. A one-to-one correspondence then ensures the existence of a mapping between abstract operators and functions in phase space.

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